

The System of R-Exp: Expanding the Algebraic Number System

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Abstract

R-exp is an alternative exponentiation-like operation based on grouped addition rather than repeated multiplication. This paper compares additive powers and roots with their conventional counterparts, examines polynomial reinterpretations, and develops a value-preserving *Quadratic Mapping*. The defining rule is kept distinct from conventional multiplication, even where the numerical results coincide. The revised zero-exponent convention is $a^{(0)} = 0$.

1 Introduction

Conventional exponentiation is based on repeated multiplication. For example, the ordinary square root and fractional power of four are related by

$$\sqrt{4} = 2, \quad 4^{1/2} = 2.$$

R-exp asks what happens when repeated addition replaces repeated multiplication as the underlying process. In this alternative operation, additive roots of negative numbers are real-valued, although they are not the same objects as conventional complex roots. The purpose of the study is to examine the resulting operations and their relationships to familiar algebraic expressions, not to identify the two exponentiation systems as equivalent.

2 An Alternate Algebraic Number System

In ordinary complex arithmetic, $i = \sqrt{-1}$ and $i^2 = -1$. In R-exp, the equation for an *additive square root* of -1 instead reads

$$x^{\langle 2 \rangle} = -1 \quad \Longleftrightarrow \quad x + x = -1,$$

so that

$$\boxed{x = -\frac{1}{2}}.$$

This does not assert that the conventional imaginary unit i equals $-\frac{1}{2}$; the two square operations have different definitions. We may denote the R-exp analogue by $i_R = -\frac{1}{2}$ when a distinct symbol is useful.

3 Additive Exponentiation Rules

3.1 Definition for a single base

For a real base a and a nonnegative integer exponent n , define

$$\boxed{a^{\langle n \rangle} = \underbrace{a + a + \cdots + a}_{n \text{ summands}}, \quad n \geq 1, \quad a^{\langle 0 \rangle} = 0.} \quad (1)$$

The angle brackets distinguish R-exp from conventional exponentiation. The number of copies of a is specified by the exponent; conventional multiplication is not used to *define* the operation.

$$\begin{aligned} a^{\langle 0 \rangle} &= 0, & a^{\langle 1 \rangle} &= a, & a^{\langle 2 \rangle} &= a + a, & a^{\langle 3 \rangle} &= a + a + a, \\ 2^{\langle 3 \rangle} &= 2 + 2 + 2 = 6, & 2^3 &= 2 \cdot 2 \cdot 2 = 8. \end{aligned}$$

3.2 Zero exponent and recurrence

The zero-exponent convention in this edition supersedes the earlier proposal $a^{(0)} = a$. The unified recurrence is

$$\boxed{a^{\langle n+1 \rangle} = a^{\langle n \rangle} + a, \quad a^{\langle 0 \rangle} = 0.} \quad (2)$$

Thus the zero exponent returns the additive identity. In particular $0^{\langle 0 \rangle} = 0$ by this R-exp definition; this makes no claim about conventional 0^0 .

3.3 Different exponents

For nonnegative integers m, n , concatenating the two groups gives

$$\boxed{a^{\langle m+n \rangle} = a^{\langle m \rangle} + a^{\langle n \rangle}.} \quad (3)$$

For instance,

$$2^{\langle 3 \rangle} + 2^{\langle 2 \rangle} = (2 + 2 + 2) + (2 + 2) = 10 = 2^{\langle 5 \rangle}.$$

The conventional *product* of these two values is different:

$$(2^{\langle 3 \rangle})(2^{\langle 2 \rangle}) = 6 \cdot 4 = 24.$$

The distinction between grouping by addition and multiplying evaluated expressions must be maintained.

3.4 Powers compared

Expression	R-exp value	Conventional value
$1^{\langle 2 \rangle}$ vs. 1^2	2	1
$1^{\langle 1 \rangle}$ vs. 1^1	1	1
$1^{\langle 0 \rangle}$ vs. 1^0	0	1
$2^{\langle 3 \rangle}$ vs. 2^3	6	8
$2^{\langle 4 \rangle}$ vs. 2^4	8	16
$7^{\langle 2 \rangle}$ vs. 7^2	14	49
$3.7^{\langle 2 \rangle}$ vs. 3.7^2	7.4	13.69

4 Additive Roots and Rational Exponents

The original paper extends the grouped-addition interpretation to reciprocal exponents by treating the additive n th root of a as the number whose sum with itself n times equals a :

$$\underbrace{x + x + \cdots + x}_{n \text{ copies}} = a \implies x = \frac{a}{n}.$$

Accordingly, for positive integers n we define

$$\boxed{a^{\langle 1/n \rangle} = \frac{a}{n}.} \quad (4)$$

This is an *extension* of the integer-count definition, not literally a fractional number of summands. More generally, for $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$,

$$\boxed{a^{\langle p/q \rangle} = \underbrace{\frac{a}{q} + \cdots + \frac{a}{q}}_{p \text{ summands}} = \frac{pa}{q}.$$

Examples of additive roots include

$$\begin{aligned} \sqrt[4]{4} &:= 4^{\langle 1/2 \rangle} = 2, & \sqrt[7]{7} &:= 7^{\langle 1/2 \rangle} = 3.5, \\ \sqrt[3]{2} \text{ (conventional)} &\approx 1.259921, & 2^{\langle 1/3 \rangle} &= \frac{2}{3}, \\ -2^{\langle 1/2 \rangle} &= -1, & 3^{\langle 1/7 \rangle} &= \frac{3}{7}. \end{aligned}$$

Here $\sqrt[a]{a}$ is an additive-root notation used only for the R-exp square-root operation; it is not the conventional radical. For example,

$$2^{\langle 1/3 \rangle \langle 3 \rangle} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2.$$

In conventional algebra, $\sqrt{-24} = 2i\sqrt{6}$, whereas the R-exp additive square root is

$$\sqrt[-24]{-24} = -24^{\langle 1/2 \rangle} = -12.$$

The different answers reflect different definitions of “square root.” Negative *bases* are therefore admissible under this additive-root rule. Negative *exponents* remain a separate question requiring an explicit extension.

5 Polynomial Equations in Conventional and R-exp Systems

A polynomial with conventional powers and an expression formed by replacing those powers with R-exp are different functions. Their roots need not coincide.

5.1 A quadratic expression

Conventionally,

$$(x - 1)^2 = x^2 - 2x + 1.$$

In R-exp,

$$(x - 1)^{\langle 2 \rangle} = (x - 1) + (x - 1) = 2x - 2.$$

This is linear in x , not a conventional quadratic polynomial.

5.2 Cubic example

For the conventional cubic $x^3 - 2x + 2 = 0$, Cardano's method produces one real root approximately $x \approx -1.769292$, together with two nonreal roots. Under R-exp reinterpretation,

$$\begin{aligned} x^{\langle 3 \rangle} - 2x + 2 &= 0, \\ 3x - 2x + 2 &= 0, \\ x &= \boxed{-2}. \end{aligned}$$

5.3 Quartic example

The conventional equation

$$x^4 - 4x^2 + 4 = 0$$

factors as $(x^2 - 2)^2 = 0$, with roots $x = \pm\sqrt{2}$. Its R-exp reinterpretation gives

$$\begin{aligned} x^{\langle 4 \rangle} - 4x^{\langle 2 \rangle} + 4 &= 0, \\ 4x - 8x + 4 &= 0, \\ x &= \boxed{1}. \end{aligned}$$

5.4 Quintic example

The conventional quintic is

$$x^5 - 5x^4 + 5x^3 - 5x^2 + 5x - 5 = 0.$$

The corresponding R-exp expression becomes

$$\begin{aligned} x^{\langle 5 \rangle} - 5x^{\langle 4 \rangle} + 5x^{\langle 3 \rangle} - 5x^{\langle 2 \rangle} + 5x - 5 &= 0, \\ 5x - 20x + 15x - 10x + 5x - 5 &= 0, \\ x &= \boxed{-1}. \end{aligned}$$

These examples demonstrate how replacing conventional powers with grouped-addition powers changes the equations rather than solving the original polynomials by a different method.

6 Quadratic Mapping

The name *Quadratic Mapping* denotes a numerical correspondence between conventional quadratic values and selected R-exp expressions. It does not redefine the R-exp operation.

Theorem 1 (Diagonal Quadratic Mapping). *For every nonnegative integer a ,*

$$\boxed{a^2 = a^{\langle a \rangle}}. \tag{5}$$

The left-hand side uses conventional exponentiation; the right-hand side uses grouped addition.

Proof. For positive integer a , $a^{\langle a \rangle}$ is the sum of exactly a copies of a . Its numerical value is therefore $a \cdot a = a^2$. When $a = 0$, the R-exp zero rule gives $0^{\langle 0 \rangle} = 0 = 0^2$. $\square$

Conventional square	R-exp expression	Common value
0^2	$0^{(0)}$	0
2^2	$2^{(2)} = 2 + 2$	4
3^2	$3^{(3)} = 3 + 3 + 3$	9
5^2	$5^{(5)}$	25
7^2	$7^{(7)}$	49
9^2	$9^{(9)}$	81

The conventional exponent is fixed at two; the R-exp exponent varies with the base. This correspondence applies to even as well as odd nonnegative integers.

6.1 Factorized quadratic expressions

For positive integers a, b, c , the conventional expansion

$$(a - b)(a - c) = a^2 - (b + c)a + bc$$

can be represented numerically by

$$(a - b)(a - c) = a^{(a)} - a^{(b+c)} + b^{(c)}. \quad (6)$$

The factorization on the left uses conventional multiplication; the right-hand side uses grouped addition in each bracketed power, followed by ordinary subtraction and addition. For $a = 7$, $b = 2$, and $c = 3$,

$$(7 - 2)(7 - 3) = 20 = 7^{(7)} - 7^{(5)} + 2^{(3)} = 49 - 35 + 6.$$

The identity extends to other domains only when the required R-exp exponents are defined. The quadratic function $Q_R(a) = a^{(a)}$ is quadratic on the nonnegative integer diagonal, although $a^{(n)}$ is linear in a for each fixed positive integer n .

6.2 Higher-power value mappings

For positive integers a, k ,

$$a^k = a^{(a^{k-1})},$$

where a^{k-1} in the R-exp exponent is evaluated conventionally. For example,

$$3^3 = 27 = 3^{(9)} = \underbrace{3 + \dots + 3}_{9 \text{ summands}}.$$

This is a numerical translation between two expressions, not an equivalence of their exponentiation laws.

7 The R-exp Analogue of i and Iteration

The original investigation considered $i_R = -\frac{1}{2}$ because

$$i_R^{(2)} = -\frac{1}{2} - \frac{1}{2} = -1.$$

Conventional repeated multiplication of the real number $-\frac{1}{2}$ yields the sequence

$$-\frac{1}{2}, \quad \frac{1}{4}, \quad -\frac{1}{8}, \quad \frac{1}{16}, \quad -\frac{1}{32}, \dots$$

and the limit

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n = 0.$$

This conventional multiplicative iteration is not the same as the R-exp expression $-1/2^{\langle n \rangle}$, which is a sum of n copies of $-\frac{1}{2}$. Maintaining this distinction is essential when comparing the two systems.

8 Conclusion

R-exp is defined by grouped addition, with $a^{\langle 0 \rangle} = 0$ and $a^{\langle 1 \rangle} = a$. Its additive recurrence and exponent-addition law hold uniformly for nonnegative integer exponents, while rational exponents can be introduced through explicitly defined additive-root extensions. Quadratic Mapping gives exact numerical correspondences between conventional squares and diagonal R-exp expressions and between certain factored quadratics and grouped-addition expressions.

These correspondences do not, by themselves, establish a new algebraic isomorphism, a Clifford algebra, or a multiplication law for split-complex, triplex, quaternionic, or octonionic numbers. Those possible extensions require independent definitions and consistency tests. The value of the investigation is in making the rules and their consequences precise.